Let's start with a very interesting and important topic that GMAT loves to test you on – Factors.

Factors are the divisors of a number. 1 is a factor of every number and the number itself is also a factor of that number. For the sake of giving the definition, let me say: A positive integer x is a factor of a positive integer N when there exists another positive integer y such that $x \times y = N$. In other words, when N is divided by x, it doesn't leave any remainder.

Now, let's cut to the chase and go on to real business.

Tell me, how many factors does 315 have and what are they? (Isn't that an ugly number! Unlike 36 or 72 or 81 – numbers easy to work with. But let's work with what we have.)

First of all, let me break it down to smaller numbers: $315 = 63 \times 5 = 7 \times 9 \times 5$ (looks better now)

Now we know that 5, 7, 9, 3 are all factors of 315 because they divide 315 completely. But there are others too e.g. 15 ($15 \times 21$ gives 315). So how do we ensure that we get ALL the factors of 315?

Let me digress here with a few questions:

Q: Is 2 a factor of 315?

A: No! 2 does not divide 315 completely.

Q: Is 9 a factor of 315?

A: Yes, we can see that it is. $9 \times 35$ gives us 315.

Q: So then 35 is also a factor of 315, isn't it?

A: Yes, because 35 will divide 315 completely and give 9 as quotient.

Similarly, if 7 is a factor of 315, we should have a corresponding factor 45 such that, $7 \times 45 = 315$. Get the picture? (Let's assume you nodded your head.)

Back to the topic at hand now. Let's retain the Q&A format.

Q: 315 has lots of factors: 1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315. How do we get all of them?

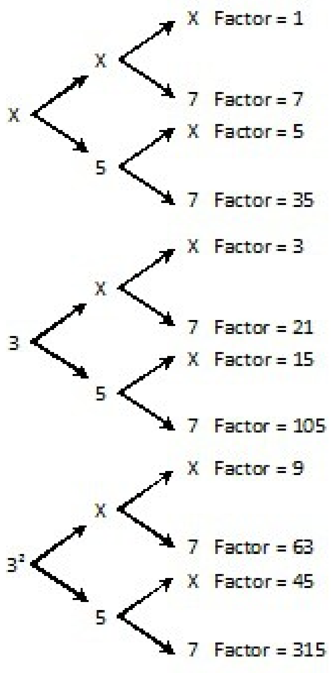
A: For that, I first need to break it down to its prime factors $315 = 3^2 \times 5 \times 7$. We get all the factors of 315 by combining the prime factors in as many different ways as we can. i.e. we take a 3 alone; it is a factor of 315. We take a $3^2$ alone; it is also a factor of 315. We take a 5 alone; it is again a factor of 315. We take a 3 and a 5 and multiply them to get 15 as a factor and so on…

Q: In how many different ways can we combine the prime factors of 315?

A: In $3 \times 2 \times 2$ ways

Q: Wait a sec! How did we get this?

A: We have to take a combination of 3, 3, 5 and 7 to make a factor. We can do this in any way we like. We can make a factor by taking a 3, a 5 and a 7. We get 105. Or we can make a factor by taking two 3s, no 5 and a 7. We get 63 as a factor. Therefore, we can choose 3 in three ways (take no 3, take one 3 or take two 3s), 5 in two ways (no 5 and one 5) and 7 in two ways (no 7 and one 7). So there are, in all, 3 × 2 × 2 = 12 ways to make a factor of 315. The diagram below illustrates this concept.




In general, **if a number can be written as N = x^p. y^q.z^r…..(where x, y, z are all distinct prime numbers), then total number of factors of N is (p + 1)(q + 1)(r + 1)…**

(the + 1 indicates the case in which you do not take that prime factor while making your factor)

Let me write down all the factors of 315 in increasing order:

1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315

Q: Do you notice a pattern? No?

Try again now

1, 3, 5, 7, 9, 15,

315, 105, 63, 45, 35, 21

Q: How about it?

A: Look at each column individually: 1 × 315 = 315; 3 × 105 = 315; 5 × 63 = 315; 7 × 45 = 315; 9 × 35 = 315; 15 × 21 = 315!

Q: Co-incidence?

A: I think not. Numbers equidistant from the left and the right multiply to give the original number. This is where the importance of the definition of factors given above comes in (You thought I was just wasting time by giving the definition, isn't it?) Every factor x (let's say 7) of a number (315 here) will have a factor y (45 here) such that x × y = N (7 × 45 = 315). The smaller the x, the greater the y to make up N. So if a factor is third smallest, it will multiply the third greatest to give you the number N.

So now, if we want to write down all the factors of 315, we just write down the small ones

1, 3, 5, 7, 9, 15

And then next to them write their corresponding big factors:

| | | | | | |
|---|---|---|---|---|---|
| 1, | 3, | 5, | 7, | 9, | 15 |
| 315, | 105, | 63, | 45, | 35, | 21 |

To ensure that we haven't missed any of the small ones, we can use the formula and find out how many total factors does the number have. Life made easy!

Something to think about:

Q: What happens if the total number of factors is odd?